Lecture 11

We have shown that

$$\overbrace{Harmonic}^{Case\ \Delta u \geq 0}\ (C^2)\ \underset{\xleftarrow{Koebe1906}}{\xrightarrow{poison1820}}\ \overbrace{Mean\ Value\ Property}^{u(y)\leq \int_{B_r(y)} u}\ (C) \xrightarrow{Riemann1851} \overbrace{Max\ Property}^{\sup_\Omega \leq \sup_{\partial\Omega}}$$

Lemma 1. *Let $u \in C^0(\Omega)$ satisfy the Mean Value Property. Then, $u \in C^1(\Omega)$ and for any $\eta \in S^{n-1}$, $\partial_\eta u$ satisfies the Mean Value Property.*

Proof. Let $y \in \Omega$ centred in a ball B_r. We wish first to assert that $u \in C^1(\Omega)$ hence we consider trajectory $y + \eta t$, η chosen from S^{n-1}. We have a similar ball $B_r(y + \eta t)$, so for a positive sufficiently small$t$, we have a perturbation the position of the sphere. By MVP assumption on u we have

$$u(y + \eta t) - u(y) = \frac{1}{|B_r|}\int_{B_r} [u(x + \eta t) - u(x)]dx$$

in particular,

$$u(y + \eta t) - u(y) = \frac{1}{|B_r|}\int\limits_{B_r(y+\eta t)} u(x + \eta t)dx - \frac{1}{|B_r|}\int\limits_{B_r(y)} u(x)dx.$$

Note that by this integration the intersections of the balls of $B_r(y)$ and $B_r(y + \eta t)$, which we denote by V, will vanish under the integral, hence we so we consider $B_r^+ = B_r(y+\eta t)/V$ and $B_r^- = B_r(y)/V$. We have

$$u(y + \eta t) - u(y) = \frac{1}{|B_r|}\int\limits_{B_r^+} u(x + \eta t)dx - \frac{1}{|B_r|}\int\limits_{B_r^-} u(x)dx$$

Let ν be unit normal to the ball and let φ the angle between the trajectory path $y + \eta t$ and the outward direction ν to the surface of B_r. The integration measure dx will be mo

$$= \frac{1}{|B_r|}\int\limits_{\partial B_r^+(y)}\int_0^t u(x + \eta s)\underbrace{\cos\varphi}_{\eta\cdot\nu}\ d^{n-1}xds - \frac{1}{|B_r|}\int\limits_{\partial B_r^-(y)}\int_0^t u(x + \eta s)(-\cos\varphi)\ d^{n-1}xds$$

$$\frac{1}{|B_r|}\int\limits_{\partial B_r(y)}\int_0^t \underbrace{u(x + \eta s)}_{u(x)+\psi(x,s)}\ \eta\cdot\nu\ d^{n-1}xds$$

$$|\psi(x, s)| \to 0 \qquad uniformly\ in\ x\ as\ s \to 0$$

(this means it converges without any dependence on x)

$$= \frac{t}{|B_r|}\int_{\partial B_r(y)} u(x)\eta\cdot\nu d^{n-1}x + o(t)$$

$$\implies \partial_\eta u(y) = \frac{1}{|B_r|}\int_{\partial B_r(y)} \overbrace{u\eta}^{F-field} \cdot\nu = \frac{1}{|B_r|}\int_{B_r(y)} \partial_\eta u$$

where we used the divergence theorem since

$$\partial_1(u\eta_1) + ... + \partial_n(u\eta_n) = \eta_1\partial_1 u + ... + \eta_n\partial_n u = \partial_\eta u$$

hence $\partial_\eta u \in C^0(\Omega)$. $\square$

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Corollary 1. *Let $u \in C^0(\Omega)$ satisfy Mean Value Property. Then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.*

Proof.

$$\Delta u(y) = \frac{1}{|B_r} \int_{B_r(y)} \vec{\nabla} \cdot \vec{\nabla} u = \frac{1}{|B_r|} \int_{\partial B_r(y)} \partial_\nu u$$

$$u(y) = \frac{1}{|S^{n-1}| r^{n-1}} \int_{S^{n-1}} u(y + r\xi) d^{n-1}\xi \; r^{n-1}$$

Note that the we are integrating from one sphere to another about the same centre infinitesimally (direction of ν) so by the mean property they would vanish but we need to show this holds formally:

$$0 = \int_{S^{n-1}} \frac{\partial u(y + r\xi)}{\partial r} d^{n-1}\xi \implies \int_{\partial B_r(y)} \partial_\nu u = 0$$

□

Derivative Estimates

Let $\Delta u = 0$.

$$|\partial_\eta u(y) \leq \frac{1}{|B_r|} \max_{\partial B_r(y)} |u| |S^{n-1}| r^{n-1}$$

Note that $|B_r| = \frac{S^{n-1}}{n} r^{n-1}$ so

$$= \frac{n}{r} \max_{\partial B_r(y)} |u|.$$

For $u \geq 0$:

$$|\partial_\eta u(y) = \leq \frac{1}{|B_r|} \int_{\underbrace{\partial B_r(y)}_{|\partial B_r| u(y)}} u = \frac{n}{r} u(y).$$

Corollary 2 (Liouville's Theorem)**.** *u harmonic in $\mathbb{R}^n$, that is bounded below or bounded above implies $u \equiv constant$.*

Proof. Let $u \geq a$ and $u \leq b$. $u - a \geq 0$ and $b - u \geq 0$ with $u \geq 0$.

$$\partial_\eta u(y) \leq \frac{n}{r} u(y), \; \forall r > 0.$$

$$\implies \partial_\eta u \equiv 0 \implies u \equiv constant.$$

□

Corollary 3. *u harmonic in Ω, $\overline{B_r(y)} \subset \Omega$. Then*

$$|\partial^\alpha u(y)| \leq |\alpha|! \left(\frac{ne}{r}\right)^{|\alpha|} \max_{B_r(y)} |u|$$

In particular, $u \in C^\omega(\Omega)$.

Proof. Take a sphere with radius r. We take a smaller sphere within the same center with radius ρ. we approximate derivatives near by (ρ away from y) by the derivatives estimates above, by reducing order till we reach the value of the function bound at r. We reduce order of derivative to $|\beta| = |\alpha| - 1$.

$$|\partial^\alpha u(y)| \leq \frac{n}{\rho} \max_{\partial B_r(y)} |\partial^\beta u|$$

$$\rho = \frac{r}{|\alpha|} \leq .. \leq \left(\frac{n}{\rho}\right)^{|\alpha|} \max_{B_r(y)} |u|.$$

$$(\frac{n}{\rho})^{|\alpha|} = (n/r)^{|\alpha|} |\alpha|^{|\alpha|}.$$

using a trick $e^x = 1 + x + ... + x^k/k!$ so we pick $k^k/k! < e^k$. Analyticity comes from $u(y+h) = \sum_{|\alpha| \geq 0} \frac{\partial^\alpha u(y)}{|\alpha|!} h^\alpha$ so we need to show the following to get convergence in the series

$$\sum \left(\frac{ne}{r}\right)^{|\alpha|} \rho^{|\alpha|} < 0$$

. choose ρ sufficiently small.

u harmonic in Ω, $\overline{B_r(y)} \subset \Omega$. Then

$$|\partial^\alpha u(y)| \leq |\alpha|! \left(\frac{ne}{r}\right)^{|\alpha|} \max_{B_r(y)} |u| \qquad \overline{B_r(y)} \subset \Omega$$

$$v(t) = \sum_{k=0}^{m} t^k/k! \cdot v^{(k)}(0) + \frac{v^{(m+1)}(\xi)}{(m+1)!} t^{m+1} \qquad \xi \in (0, t)$$

Take $v \in C^{m+1}(a, b)$, $0, t \in (a, l)$ e.g

$v(t) = e^t$, $R_m = \frac{e^\xi}{(m+1)!} t^{m+1} \leq e^t \frac{t^{m+1}}{m+1} \to 0$ as $n \to \infty$ in the case $e^{1/x}$ we have $R_m \approx \frac{e^{-1/\xi}}{(m+1)!} (\frac{x}{\xi})^{m+1}$ depends on

Suppose

$$u \in C^\infty(\mathbb{R}^n) \qquad v(t) = u(xt). \qquad x \in \mathbb{R}^n \; t \in \mathbb{R}.$$

$$v'(t) = \partial_j (xt) x_j \qquad v''(t) = \partial_j \partial_i u(xt) x_j x_i.$$

$$...$$

$$v^{(k)}(0) = \sum_{j_1=1}^{n} ... \sum_{j_k=1}^{n} \partial_{j1} ... \partial_{jk} u(0) x_{j1} ... x_{jk}$$

$$= \sum_{|\alpha|=k} \underbrace{\frac{|\alpha|!}{\alpha_1! ... \alpha_n!}}_{\alpha!} \underbrace{\partial_1^{\alpha_1} ... \partial_n^{\alpha_n} u(0)}_{\partial^\alpha u(0)} \underbrace{x_1^{\alpha_1} ... x_n^{\alpha_n}}_{x^\alpha}$$

hence for $t = 1$

$$u(x) = \sum_{|\alpha| \leq m} \frac{\partial^\alpha u(0)}{\alpha!} x^\alpha + \sum_{|\alpha|=m+1} \frac{\partial^\alpha u(x\xi)}{\alpha!} x^\alpha \xi^{|\alpha|}$$

$$u(x) = \sum_{|\alpha|\leq m} \frac{\partial^\alpha u(z)}{\alpha!}(x-z)^\alpha + \underbrace{\sum_{|\alpha|=m+1} \frac{(y-z)^\alpha}{\alpha!}\partial^\alpha u(y-z)\partial^\alpha u(y)}_{R_m} \qquad y = \xi(x-z)+z$$

$\overline{B_R(z)} \subset \Omega$ and noting that $|\alpha|! \leq \alpha! n^{|\alpha|}$,

$$|\partial^\alpha u(y)| \leq \alpha! n^{|\alpha|}(ne/r)^{|\alpha|} \max_{B_R(z)} |u|$$

$$R_m \leq (m+1)\rho^{|\alpha|} n^{|\alpha|}(ne/r)^{|\alpha|} M = M(m+1)(n^2 e\rho/r)^{|\alpha|\to m+1}$$

$n^2 e\rho < r = R - \rho \implies \rho < R/(1+n^2 e)$ (ball within ball radius R and rho) radius of analycity draw

$\square$